Mathematical Modeling of Spacecraft Gyroscope Noise

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Hope This helps

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Gyro Noise Descriptions

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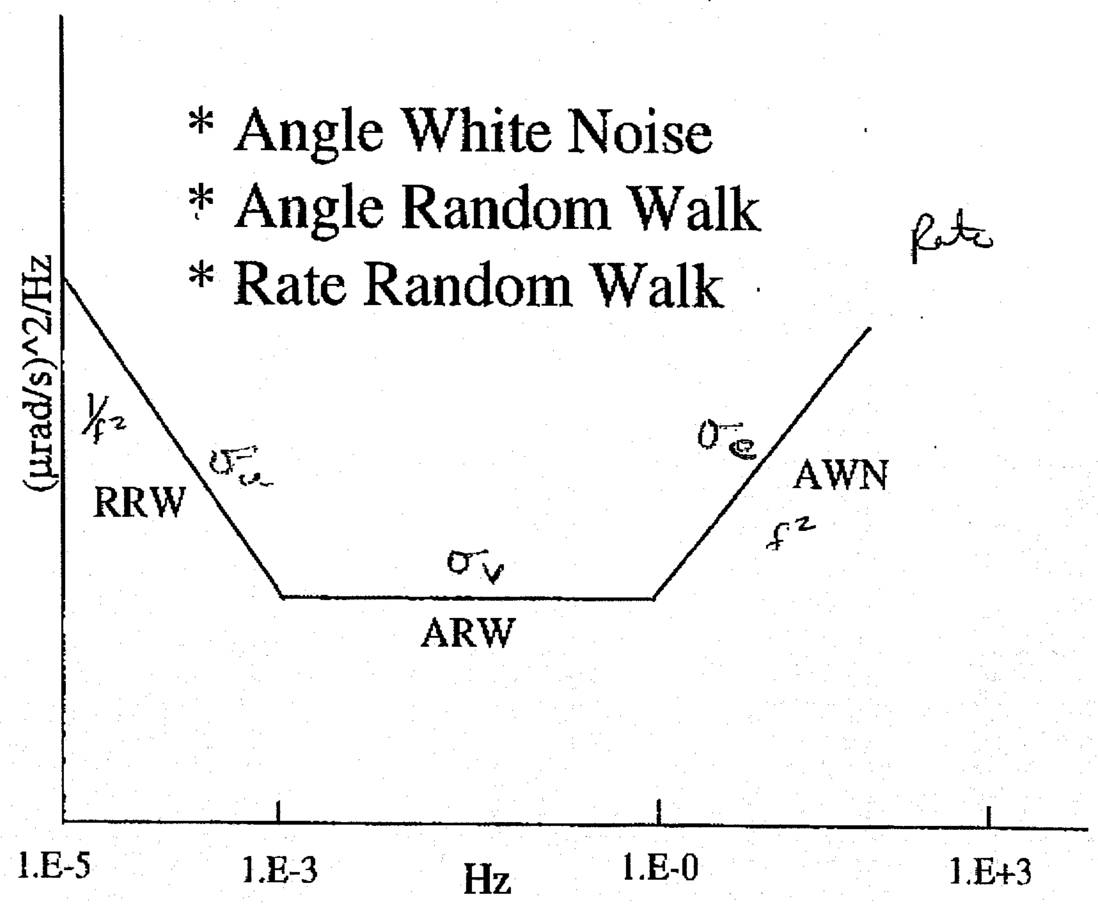
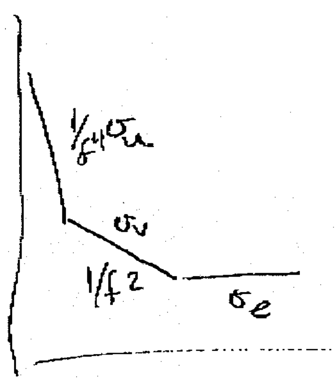
Rate Power Spectral Density Chart



Angle PSD



Modeling Gyro Noise 2/12/26

For high-accuracy applications, most strapdown gyros are rate-integrating gyros (RIGs), that is, they measure (ideally) an integrated angle

$$\theta = \int_{t_0}^{t_0+\Delta t} \omega_{input}(t)\,dt \quad , \text{where}$$

$\omega_{input}(t) = \vec{\omega} \cdot \vec{u}_{input}$ = component of the spacecraft angular velocity along an input axis represented by the unit vector $\vec{u}_{input}$, perpendicular to the gyro spin axis.

In general, $\vec{u}_{input} = A_{misalignment}\,\vec{u}_{nominal}$.

A rate-integrating gyro produces its output by digitally summing the rebalance torques needed to keep the rotor orientation fixed in the spacecraft as the spacecraft rotates.

A rate gyro measures ω_{input} rather than θ.

As we have seen, a gyro is a very complex coupled electromechanical system, and modeling noise sources in gyros from first principles would be very difficult. Fortunately, there is a fairly tractable model that contains a fair approximation to the most important effects. This model was developed by TRW engineers for the HEAO-A (High Energy Astronomy Observatory Spacecraft), most notably Robert L. Farrenkopf. HEAO-A used single-axis floated gyros but we use the same model for two-axis dry tuned gyros. This ignores errors arising from the coupling of the two axes of the two-axis gyros.

The model is

$$\dot{\theta}(t) = \omega_{input}(t) + b(t) + \eta_1(t)$$

where b is the gyro drift, with time dependence

$$\dot{b}(t) = \eta_2(t)$$

The quantities $\eta_1(t)$ and $\eta_2(t)$ are zero-mean white noise processes, which are not quite the same as the random variables we considered before, because they are functions of time. Zero-mean, as before, means that $E\{\eta_1(t)\} = E\{\eta_2(t)\} = 0$ for all t, where $E\{\ldots\}$ means expectation value.

The drift $b(t)$, whose time derivative is purely white noise, undergoes a random walk. This is mathematically the same as Brownian motion, and is sometimes also called the drunkard's walk.

We need to know about expectation value of product of η_1 and η_2. We assume that η_1 and η_2 are uncorrelated, which means that

$$E\{\eta_1(t)\,\eta_2(t')\} = 0 \quad \text{for any } t, t'.$$

Also the "white noise" approximation on η_1 and η_2 means that the value of each function at a given time is uncorrelated with the value of the same function at a different time. Thus

$$E\{\eta_1(t)\,\eta_1(t')\} = \sigma_v^2\,\delta(t-t') \quad \text{add } \sigma_e$$
$$E\{\eta_2(t)\,\eta_2(t')\} = \sigma_u^2\,\delta(t-t')$$

where $\delta(x)$ is a function that is zero for $x \neq 0$. The white noise assumption allows σ_v^2 and σ_u^2 to be functions of time, be we'll assume for simplicity that they're constant.

We haven't said anything about the value of $\delta(x)$ at $x=0$. We want it to be infinite there, so infinite that

$$\int_a^b \delta(x-y)\,dx = 1 \quad \text{if} \quad a<y<b$$

Mathematically, such a function doesn't exist, but it can be understood rigorously as a generalized function or a distribution. Physicists call $\delta(x)$ the Dirac delta function and engineers call it the unit impulse function. For real gyros we have, instead of $\delta(x)$, a function that has a narrow high peak at $x=0$; this is colored noise. White noise is easier to analyze.

For computations below we need to know $\int_0^a \int_0^b \delta(x-y)\,dx\,dy$

If $b>a$ we integrate over x first using $\int_0^b \delta(x-y)\,dx = 1$, since $0<y<a<b$

If $a>b$ we integrate over y first, using $\int_0^a \delta(x-y)\,dy = 1$, since $0<x<b<a$

We are left with the integral of 1 over a range from 0 to the minimum of a and b. Thus

$$\int_0^a \int_0^b \delta(x-y)\,dx\,dy = \text{minimum of } (a,b).$$

Now back to the gyros. From the $\dot{b}$ equation

$$b = b_0 + \int_0^{\Delta t} \eta_2(t)\,dt$$

where we've taken $t_0 = 0$ as the lower limit of integration to simplify the notation.

Then $E\{b\} = b_0$ since $E\{\eta_2\} = 0$.

$$b^2 = b_0^2 + 2b_0 \int_0^{\Delta t} \eta_2(t)\,dt + \int_0^{\Delta t} \eta_2(t)\,dt \int_0^{\Delta t} \eta_2(t')\,dt'$$

$$E\{b^2\} = b_0^2 + E\left\{\int_0^{\Delta t} \eta_2(t)\,dt \int_0^{\Delta t} \eta_2(t')\,dt'\right\}$$

$$= b_0^2 + \sigma_u^2 \int_0^{\Delta t}\int_0^{\Delta t} \delta(t-t')\,dt\,dt' = b_0^2 + \sigma_u^2 \Delta t$$

To model the drift in a simulation, we can take

$$b = b_0 + \sigma_u (\Delta t)^{1/2} \xi_b$$

where ξ_b is a gaussian-distributed random variable with mean 0 and standard deviation unity. It can be seen that this gives the correct expectation values $E\{b\}$ and $E\{b^2\}$.

The gyro output is given by

$$\Theta = \Theta_0 + \int_0^{\Delta t} (\omega_{input} + b + \eta_1)\, dt = \Theta_0 + \int_0^{\Delta t} \omega_{input}\, dt + \int_0^{\Delta t} \left[b_0 + \int_0^{t} \eta_2(\tau)\, d\tau \right] dt + \int_0^{\Delta t} \eta_1(t)\, dt$$

$$= \Theta_{perfect} + b_0 \Delta t + \int_0^{\Delta t}\int_0^{t} \eta_2(\tau)\, d\tau\, dt + \int_0^{\Delta t} \eta_1(t)\, dt$$

where $\Theta_{perfect} \equiv \Theta_0 + \int_0^{\Delta t} \omega_{input}\, dt$ is the value of the gyro output in the absence of drifts and noise.

$$E\{\Theta\} = \Theta_{perfect} + b_0 \Delta t$$

$$E\{b\Theta\} = E\left\{ \left[b_0 + \int_0^{\Delta t} \eta_2(t)\, dt \right] \left[\Theta_{perfect} + b_0 \Delta t + \int_0^{\Delta t}\int_0^{t} \eta_2(\tau)\, d\tau\, dt + \int_0^{\Delta t} \eta_1(t)\, dt \right] \right\}$$

$$= b_0 (\Theta_{perfect} + b_0 \Delta t) + E\left\{ \int_0^{\Delta t} \eta_2(t')\, dt' \int_0^{\Delta t}\int_0^{t} \eta_2(\tau)\, d\tau\, dt \right\}$$

$$= b_0 (\Theta_{perfect} + b_0 \Delta t) + \sigma_u^2 \int_0^{\Delta t}\int_0^{\Delta t}\int_0^{t} \delta(t' - \tau)\, d\tau\, dt'\, dt$$

$$= b_0 (\Theta_{perfect} + b_0 \Delta t) + \sigma_u^2 \int_0^{\Delta t} t\, dt = b_0 (\Theta_{perfect} + b_0 \Delta t) + \sigma_u^2 (\Delta t)^2 / 2$$

Since $E\{\xi_b\} = 0$ and $E\{\xi_b^2\} = 1$, these equations are satisfied if we model Θ in a simulation as

$$\Theta = \Theta_{perfect} + b_0 \Delta t + (1/2)\sigma_u \Delta t^{3/2} \xi_b + Z = \Theta_{perfect} + (b_0 + b)\Delta t / 2 + Z$$

where Z is a random variable with zero mean that is uncorrelated with ξ_b

If we get $E\{\Theta^2\}$ right, we've got all the expectation values of things linear and quadratic in Θ and b.

$$E\{\Theta^2\} = (\Theta_{perfect} + b_o \Delta t)^2 + E\left\{\int_0^{\Delta t}\int_0^{t} \eta_2(\tau)d\tau dt \int_0^{\Delta t}\int_0^{t'} \eta_2(\tau')d\tau' dt'\right\}$$

$$+ E\left\{\int_0^{\Delta t} \eta_1(t)dt \int_0^{\Delta t} \eta_1(t')dt'\right\}$$

since $E\{\eta_1\} = E\{\eta_2\} = E\{\eta_1\eta_2\} = 0$

$$E\{\Theta^2\} = (\Theta_{perfect} + b_o \Delta t)^2 + \sigma_u^2 \int_0^{\Delta t}\int_0^{\Delta t}\int_0^{t}\int_0^{t'} \delta(\tau-\tau')\, d\tau' d\tau\, dt' dt$$
$$+ \sigma_v^2 \int_0^{\Delta t}\int_0^{\Delta t} \delta(t-t')\, dt\, dt'$$

$$= (\Theta_{perfect} + b_o \Delta t)^2 + \sigma_u^2 \int_0^{\Delta t}\int_0^{\Delta t} [\text{minimum of } (t,t')]\, dt' dt + \sigma_v^2 \Delta t$$

The remaining integral can be split into parts with $t'<t$ and $t'>t$.

$$\int_0^{\Delta t}\left[\int_0^{t} t' dt' + \int_t^{\Delta t} t\, dt'\right] dt = \int_0^{\Delta t} \left[t^2/2 + t(\Delta t - t)\right] dt$$

$$= \Delta t \int_0^{\Delta t} t\, dt - (1/2)\int_0^{\Delta t} t^2 dt = \Delta t^3 [(1/2) - (1/6)] = \Delta t^3/3$$

Thus $E\{\Theta^2\} = (\Theta_{perfect} + b_o \Delta t)^2 + \sigma_u^2 (\Delta t)^3/3 + \sigma_v^2 \Delta t$.

But our equation for modeling Θ in a simulation gives

$$E\{\Theta^2\} = (\Theta_{perfect} + b_o \Delta t)^2 + \sigma_u^2 (\Delta t)^3/4 + E\{Z^2\}$$

Comparing these shows that we must have

$$E\{Z^2\} = \sigma_v^2 \Delta t + \sigma_u^2 (\Delta t)^3/12$$

This can be accomplished by putting

$$Z = \sqrt{\sigma_v^2 \Delta t + \sigma_u^2 (\Delta t)^3/12}\ \xi_\theta$$

where ξ_θ is a gaussian-distributed random variable with zero mean and standard deviation unity, uncorrelated with ξ_b.

Thus we need two calls to a Gaussian random-number generator to get ξ_b and ξ_θ, after which

$$b = b_0 + \sigma_u(\Delta t)^{1/2}\xi_b \rightarrow \theta = \theta_{perf} + b_0\Delta t + \sqrt{\frac{\sigma_u^2\Delta t^3}{4}} + \sqrt{\sigma_v^2\Delta t + \sigma_u^2(\Delta t)^3/12 + \sigma_e^2}$$

$$\Theta = \Theta_{perfect} + (b_0+b)\Delta t/2 + \sqrt{\sigma_v^2\Delta t + \sigma_u^2(\Delta t)^3/12 + \sigma_e^2}\,\xi_\theta$$

We encode Θ for output, including scale factor error, output noise, and quantization, if we wish. Then b and Θ become b_0 and Θ_0 for the next step. If the standard deviation of the output noise is denoted by σ_e, we recover Fallon's result, equation (7-143) in Wertz.

If the rates are slowly varying we can use the trapezoid approximation to integrate $\Theta_{perfect}$

$$\Theta_{perfect} = \Theta_0 + \int_0^{\Delta t} \omega_{input}\,dt \simeq \Theta_0 + (\omega_{input\,0} + \omega_{input})\Delta t/2$$, giving

$$\Theta = \Theta_0 + \left[(\omega_{input}+b)_0 + (\omega_{input}+b)\right]\Delta t/2 + \sqrt{\sigma_v^2\Delta t + \sigma_u^2\Delta t^3/12}\,\xi_\theta$$

For some rate-integrating gyro packages (DRIRU II, for example) the electronics computes and outputs an analog rate

$$\omega_{gyro} = \frac{\Theta - \Theta_0}{\Delta t} = \left[(\omega_{input}+b)_0 + (\omega_{input}+b)\right]/2 + \sqrt{\sigma_v^2/\Delta t + \sigma_u^2\Delta t/12}\,\xi_\theta$$

Most precise ACS use the integrated output Θ rather than ω_{gyro}, however. The analog rate signal mode is needed by analog "safe hold" control systems.

The expression for ω_{gyro} is correct if the Δt at which the gyro electronics updates the rate output is the same as the Δt of the simulation. It could be (and often is) that the cycle time of the gyro electronics, which we shall call τ, is much less than the OBC Δt. In this case

$$\omega_{gyro} = \left[(\omega_{input} + b)' + (\omega_{input} + b)\right]/2 + \sqrt{\sigma_v^2/\tau + \sigma_u^2\tau/12}\ \xi_x$$

where the prime indicates time $t-\tau$, and ξ_x is some other zero mean gaussian random variable with standard deviation 1. We don't know the quantities at $t-\tau$, so we use

$\omega_{input}' \approx \omega_{input}$

$b' = b + \sigma_u \tau^{1/2} \xi_y$

Then

$$\omega_{gyro} = \omega_{input} + b + \left(\tfrac{1}{2}\sigma_u \tau^{1/2} \xi_y + \sqrt{\sigma_v^2/\tau + \sigma_u^2\tau/12}\ \xi_x\right)$$

The quantity in parentheses is zero mean random noise with variance (assuming ξ_x and ξ_y are uncorrelated)

$$\text{variance} = \frac{\sigma_u^2\tau}{4} + \frac{\sigma_v^2}{\tau} + \frac{\sigma_u^2\tau}{12} = \frac{\sigma_v^2}{\tau} + \frac{\sigma_u^2\tau}{3}.$$

Thus we can write

$$\omega_{gyro} = \omega_{input} + b + \sqrt{\sigma_v^2/\tau + \sigma_u^2\tau/3}\ \xi_\omega$$

In this case, we need three random number calls, for σ_b, σ_θ, and σ_ω.

Minimizing the noise term wrt τ gives $\tau = \sqrt{3}\,\sigma_v/\sigma_u$ and $\omega_{gyro} = \omega_{input} + b + \sqrt{2\sigma_u\sigma_v/\sqrt{3}}\ \xi_\omega$

For Farrenkopf's example (JGC V. 1, no. 4)

$\sigma_u = 4.81 \times 10^{-5}\ \widehat{sec}/s^{3/2}$, $\sigma_v = 0.200\ \widehat{sec}/s^{1/2}$

$\tau_{optimal} = \dfrac{\sqrt{3} \times .2}{4.81 \times 10^{-5}} = 7202\ sec$

In practice τ is much less than this and σ_v must be the dominant error term. Say $\tau = 32$ msec, then

$\sigma_v^2/\tau + \sigma_u^2 \tau/3 = (1.25 + 2.5 \times 10^{-11})(\widehat{sec}/s)^2$

so $\sqrt{\sigma_v^2/\tau + \sigma_u^2 \tau/3} \simeq \sqrt{1.25}\ \widehat{sec}/s = 1.12\ \widehat{sec}/s$ ($\tau = 32$ ms

$= 0.56$ " ($\tau = 128$ ms

$= 0.28$ " ($\tau = 512$ ms

$1\ \widehat{sec}/s = 1$ deg/hour, which is OK for control